

Anomalies in the ERG approach

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The antifield formalism adapted in the exact renormalization group is found to be useful for describing a system with some symmetry, especially the gauge symmetry. In the formalism, the vanishing of the quantum master operator implies the presence of a symmetry. The QM operator satisfies a simple algebraic relation that will be shown to be related to the Wess-Zumino condition for anomalies. We also explain how an anomaly contributes to the QM operator.

§1. Introduction

In the exact renormalization group, it often happens that the symmetry of a system is not compatible with the momentum cutoff. This is particularly important for a gauge theory since we do not have a convenient way of regularizing the theory without breaking the gauge symmetry.^{*)}

As shown in earlier works,^{2),3)} any symmetry survives even after introducing the momentum cutoff Λ . As the cutoff changes, the Wilson action and the symmetry transformation change in their appearance. Still, we may write the Ward-Takahashi (WT) identity $\Sigma_\Lambda = 0$, which may be elevated to the quantum master equation (QME) $\bar{\Sigma}_\Lambda = 0$ of the Batalin-Vilkovisky (BV) antifield formalism.⁴⁾ The QME implies the presence of the symmetry in the system. The QME $\bar{\Sigma}_\Lambda = 0$ in the limit of $\Lambda \rightarrow 0$ is found to be equivalent to the Zinn-Justin equation.^{**)} Since both equations are manifestations of the presence of a symmetry, this correspondence is quite natural.

With antifields, we introduce a canonical structure that has been fully utilized in its application to the ERG. In later sections, we find an algebraic relation for $\bar{\Sigma}_\Lambda$ derived from its definition and the canonical structure. Since this is an algebraic relation, it holds even if $\bar{\Sigma}_\Lambda \neq 0$, i.e., even in the absence of the corresponding symmetry. Naturally, we expect that the effective action also satisfies some algebraic condition similar to the one for the QM operator. Actually, we already know such a condition, that is, the Wess-Zumino (WZ) condition. Therefore, in the case of $\bar{\Sigma}_\Lambda \neq 0$, the QM operator must be related to an anomaly. It is the subject of the present paper to explain how the QM operator is related to an anomaly.^{***)}

We will see that the QM operator is a composite operator, essentially an anomaly, that flows under the change of the cutoff scale. As a composite operator, it changes the expression. In the limit of $\Lambda \rightarrow 0$, it is related to the well-known anomaly written for the effective action. The expression of the composite operator also simplifies in the other limit of $\Lambda \rightarrow \infty$: the QM operator becomes an anomaly times a ghost factor. This will be shown explicitly for an abelian theory.

The paper is organized as follows. In the next section, we describe some results reported earlier,⁵⁾ which are needed for later discussion. In this paper, we follow the notations used in the review article. Via the relation to the 1PI counter object of the QM operator, we find how the QM operator tends to the anomaly in the $\Lambda \rightarrow 0$ limit. In §3 and §4, we study the other limit of $\Lambda \rightarrow \infty$ for an abelian gauge theory coupled to massless fermions. We will see that the same result is obtained by two different methods. The last section is devoted to summary and discussion. There, we point out the relation of the Wess-Zumino condition to the algebraic condition on the QM operator. The proof of the relation is given in the Appendix.

^{*)} A regularization procedure to respect the gauge symmetry has been proposed. See Ref. 1) and references therein.

^{**)} See §10 and Appendix D of Ref. 5).

^{***)} Although we consider gauge anomalies in this paper, our discussion may be extended to global anomalies that also have been studied in the BV formalism.^{6),7)}

§2. Antifield formalism and its application to ERG

Here, we describe some results that will be useful to understand later discussion.

Given a classical gauge fixed action $S_{cl}[\phi]$ for a generic gauge theory, we may write an extended action

$$\bar{S}_{cl}[\phi, \phi^*] \equiv S_{cl}[\phi] + \phi_A^* \delta \phi^A. \quad (2.1)$$

The field ϕ^A represents the gauge, ghost, antighost, auxiliary fields as well as possible matter fields. The BRST transformation is denoted as $\delta \phi^A$. ϕ_A^* represents the corresponding antifield with the opposite Grassmann parity to that of ϕ^A .

In the space of ϕ^A and ϕ_A^* , we define the canonical structure via an antibracket: for any field variables X and Y , we define

$$(X, Y) \equiv \frac{\partial^r X}{\partial \phi^A} \frac{\partial^l Y}{\partial \phi_A^*} - \frac{\partial^r X}{\partial \phi_A^*} \frac{\partial^l Y}{\partial \phi^A}. \quad (2.2)$$

Following the definitions (2.1) and (2.2), we obtain

$$(\bar{S}_{cl}, \bar{S}_{cl}) = 2(\delta S_{cl} + \phi_A^* \delta^2 \phi^A). \quad (2.3)$$

The r.h.s. of (2.3) vanishes if the action is BRST invariant and the transformation is nilpotent. Namely, under these two conditions, the action \bar{S}_{cl} satisfies the classical master equation (CME): $(\bar{S}_{cl}, \bar{S}_{cl}) = 0$.

We now generalize the above consideration. Let $\bar{S}[\phi, \phi^*]$ be an action that defines a quantum system via the functional integration over ϕ . Under the BRST transformation of fields

$$\delta \phi^A = \frac{\partial^l \bar{S}}{\partial \phi_A^*},$$

the changes in the action and the functional measure are summed up to the quantum master operator:

$$\bar{\Sigma}[\phi, \phi^*] \equiv \frac{\partial^r \bar{S}}{\partial \phi^A} \frac{\partial^l \bar{S}}{\partial \phi_A^*} + \frac{\partial^r}{\partial \phi^A} \delta \phi^A = \frac{1}{2}(\bar{S}, \bar{S}) + \Delta \bar{S}, \quad (2.4)$$

where we define

$$\Delta \equiv (-)^{\epsilon_A+1} \frac{\partial^r}{\partial \phi^A} \frac{\partial^r}{\partial \phi_A^*}. \quad (2.5)$$

The system is BRST invariant quantum mechanically if the two contributions cancel:

$$\bar{\Sigma}[\phi, \phi^*] = 0. \quad (2.6)$$

We call this equation the quantum master equation (QME).

We define the quantum BRST transformation as

$$\delta_Q X \equiv (X, \bar{S}) + \Delta X \quad (2.7)$$

for an arbitrary variable X . Without assuming QME, we obtain two important algebraic identities:

$$\delta_Q \bar{\Sigma}[\phi, \phi^*] = 0, \quad (2.8)$$

$$\delta_Q^2 X = (X, \bar{\Sigma}[\phi, \phi^*]). \quad (2.9)$$

These are consequences of the definitions of the quantum master operator (2.4) and the quantum BRST transformation (2.7). The identity (2.8) is crucial for the perturbative construction of symmetric theories, as shown in §7 and §8 in Ref. 5). Equation (2.9) implies that the quantum BRST transformation (2.7) is nilpotent if and only if QME (2.6) holds.

2.1. Application to ERG

For the application of BV formalism to the exact renormalization group (ERG), we take the action defined at some ultraviolet scale Λ_0 :

$$S_B[\phi] \equiv -\frac{1}{2}\phi \cdot K_0^{-1}D \cdot \phi + S_{I,B}[\phi]. \quad (2.10)$$

Here, the momentum of the propagating mode is restricted as $p^2 < \Lambda_0^2$ with a positive function $K_0(p) \equiv \kappa(p^2/\Lambda_0^2)$; the function κ behaves as

$$\kappa(p^2/\Lambda^2) \sim \begin{cases} 1, & (p^2/\Lambda^2 < 1) \\ 0, & (p^2/\Lambda^2 > 1) \end{cases} \quad (2.11)$$

We also use the following notation:

$$\phi \cdot K_0^{-1}D \cdot \phi \equiv \int_p \phi^A(-p) \frac{D_{AB}(p)}{K_0(p)} \phi^B(p).$$

Suppose that we have the extended action $\bar{S}_B[\phi, \phi^*]$ based on the BRST invariance of the bare action (2.10). Then we define the partition function as

$$\bar{Z}_B[J, \phi^*] \equiv \exp[\bar{W}_B[J, \phi^*]] \equiv \int \mathcal{D}\phi \exp(\bar{S}_B[\phi, \phi^*] + K_0^{-1}J \cdot \phi). \quad (2.12)$$

By introducing the momentum cutoff Λ , lower than the UV cutoff, we may perform momentum integration for $\Lambda^2 < p^2 < \Lambda_0^2$. This gives the Wilson action \bar{S}_Λ with the cutoff Λ and the corresponding generating functional,

$$\bar{Z}_\Lambda[J, \Phi^*] = \int \mathcal{D}\Phi \exp(\bar{S}_\Lambda[\Phi, \Phi^*] + K^{-1}J \cdot \Phi), \quad (2.13)$$

where $K(p) \equiv \kappa(p^2/\Lambda^2)$. The field Φ^A carries momentum lower than the scale Λ and we have rescaled the antifields as

$$K\Phi_A^* = K_0\phi_A^*, \quad (2.14)$$

in order to keep the canonical structure. The Wilson action takes the form,

$$\begin{aligned} \bar{S}_\Lambda[\Phi, \Phi^*] &\equiv -\frac{1}{2}\Phi \cdot K^{-1}D \cdot \Phi + \bar{S}_{I,\Lambda}[\Phi, \Phi^*], \\ \exp(\bar{S}_{I,\Lambda}[\Phi, \Phi^*]) &= \int \mathcal{D}\chi \exp\left[-\frac{1}{2}\chi \cdot (K_0 - K)^{-1}D \cdot \chi + \bar{S}_{I,B}[\Phi + \chi, \phi^*]\right]. \end{aligned} \quad (2.15)$$

Two generating functionals (2.12) and (2.13) are related as

$$\bar{Z}_B[J, \phi^*] = N_J \bar{Z}_\Lambda[J, \Phi^*], \quad (2.16)$$

where

$$\ln N_J = -\frac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} (K_0 - K) (D^{-1})^{AB} J_B. \quad (2.17)$$

In this manner, we may observe the change in the Wilson action under the change in the cutoff scale Λ . We denote QM operators at the scales Λ_0 and Λ as $\bar{\Sigma}_B$ and $\bar{\Sigma}_\Lambda$ respectively.

Rather than following the above-mentioned standard procedure, we may take a different way to integrate over the same momentum modes and introduce the effective average action. We consider the path integral

$$\exp[\bar{W}_{B,\Lambda}[J, \phi^*]] \equiv \int \mathcal{D}\phi \exp(\bar{S}_{B,\Lambda}[\phi, \phi^*] + K_0^{-1}J \cdot \phi). \quad (2.18)$$

In two path integrals, (2·12) and (2·18), the action $\bar{S}_{B,\Lambda}$ differs from \bar{S}_B only in the kinetic term: the action $\bar{S}_{B,\Lambda}$ has the kinetic term

$$\phi \cdot (K_0 - K)^{-1} D \cdot \phi , \quad (2·19)$$

and two actions are related as

$$\bar{S}_B = \bar{S}_{B,\Lambda} + \frac{1}{2} \phi \cdot R_\Lambda \cdot \phi \quad (2·20)$$

where

$$[R_\Lambda(p)]_{BA} \equiv D_{BA}(p) \left(\frac{1}{K_0 - K} - \frac{1}{K_0} \right) . \quad (2·21)$$

In particular, the two actions become the same in $\Lambda \rightarrow 0$.

Since the factor $K_0 - K \sim 1$ for $\Lambda^2 < p^2 < \Lambda_0^2$, while it is zero otherwise, this kinetic term allows only the modes with $\Lambda^2 < p^2 < \Lambda_0^2$ to contribute to the path integral. From the generating functional defined in Eq. (2·18), we define the effective average action as

$$\bar{\Gamma}_{B,\Lambda}[\varphi_\Lambda, \phi^*] \equiv \bar{W}_{B,\Lambda}[J, \phi^*] - K_0^{-1} J \cdot \varphi_\Lambda , \quad (2·22)$$

where

$$K_0^{-1} \varphi_\Lambda(p) \equiv \frac{\partial^l \bar{W}_{B,\Lambda}[J, \phi^*]}{\partial J(-p)} . \quad (2·23)$$

It is the Wetterich equation, the flow of the effective average action, that is often used for practical calculations.

In the limit of $\Lambda \rightarrow 0$, the path integral of the r.h.s. of Eq. (2·18) reduces to that of Eq. (2·12). Therefore, the effective average action is nothing but the ordinary effective action to be denoted as $\bar{\Gamma}_B[\varphi, \phi^*]$, where $\varphi \equiv \lim_{\Lambda \rightarrow 0} \varphi_\Lambda$: namely,

$$\lim_{\Lambda \rightarrow 0} \bar{\Gamma}_{B,\Lambda}[\varphi_\Lambda, \phi^*] = \bar{\Gamma}_B[\varphi, \phi^*] . \quad (2·24)$$

2.2. QME and Zinn-Justin equation

Let us introduce the path integral average of the QM operator $\bar{\Sigma}_B[\phi, \phi^*]$

$$\bar{\Sigma}_{B,\Lambda}^{1PI}[\varphi_\Lambda, \phi^*] \equiv \exp[-\bar{W}_{B,\Lambda}[J, \phi^*]] \int \mathcal{D}\phi \bar{\Sigma}_B[\phi, \phi^*] \exp\left(\bar{S}_{B,\Lambda}[\phi, \phi^*] + K_0^{-1} J \cdot \phi\right) . \quad (2·25)$$

Further rewriting the r.h.s. of Eq. (2·25) in terms of the effective average action, we find

$$\begin{aligned} \bar{\Sigma}_{B,\Lambda}^{1PI}[\varphi_\Lambda, \phi^*] &= \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^A} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} \\ &\quad + [R_\Lambda]_{BA} \left(-(\bar{\Gamma}^{(2)})_{B,\Lambda}^{-1} \frac{\partial^l}{\partial \varphi_\Lambda^C} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} + \varphi_\Lambda^B \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} \right) . \end{aligned} \quad (2·26)$$

Since $R_\Lambda \rightarrow 0$ (cf. Eq. (2·21)) in the limit of $\Lambda \rightarrow 0$, we find

$$\bar{\Sigma}_B^{1PI} \equiv \lim_{\Lambda \rightarrow 0} \bar{\Sigma}_{B,\Lambda}^{1PI} = \frac{\partial^r \bar{\Gamma}_B}{\partial \varphi^A} \frac{\partial^l \bar{\Gamma}_B}{\partial \phi_A^*} . \quad (2·27)$$

The vanishing of the quantum master operator $\bar{\Sigma}_B[\phi, \phi^*] = 0$ implies the presence of a symmetry. Via (2·25), this corresponds to the modified Slavnov-Taylor identity⁸⁾ $\bar{\Sigma}_{B,\Lambda}^{1PI} = 0$, which reduces to the Zinn-Justin equation for the effective action $\bar{\Gamma}_B$ in the limit of $\Lambda \rightarrow 0$.

2.3. Flow equation and composite operator

Under the scale change, the Wilson action changes according to the Polchinski equation⁹⁾

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} \bar{S}_\Lambda &= \int_p (K^{-1} \Delta)(p) \left[\Phi^A(p) \frac{\partial^l \bar{S}_\Lambda}{\partial \Phi^A(p)} - \Phi_A^*(p) \frac{\partial^l \bar{S}_\Lambda}{\partial \Phi_A^*(p)} \right] \\ &+ \frac{1}{2} \int_p (-)^{\epsilon_A} (D^{-1} \Delta)^{AB}(p) \left[\frac{\partial^l \bar{S}_\Lambda}{\partial \Phi^B(-p)} \frac{\partial^r \bar{S}_\Lambda}{\partial \Phi^A(p)} + \frac{\partial^l \partial^r \bar{S}_\Lambda}{\partial \Phi^B(-p) \partial \Phi^A(p)} \right], \end{aligned} \quad (2.28)$$

where

$$\Delta(p^2/\Lambda^2) \equiv \Lambda \frac{\partial}{\partial \Lambda} \kappa(p^2/\Lambda^2). \quad (2.29)$$

Together with the boundary condition

$$\bar{S}_\Lambda|_{\Lambda=\Lambda_0} = \bar{S}_B, \quad (2.30)$$

the flow equation (2.28) determines the Wilson action uniquely.

We define a composite operator $\bar{\mathcal{O}}_\Lambda$ as a functional for which \bar{S}_Λ and its infinitesimal perturbation $\bar{S}_\Lambda + \epsilon \bar{\mathcal{O}}_\Lambda$ satisfy the same flow equation (2.28). The flow of such an operator is given as

$$-\Lambda \frac{\partial}{\partial \Lambda} \bar{\mathcal{O}}_\Lambda = \bar{\mathcal{D}} \bar{\mathcal{O}}_\Lambda, \quad (2.31)$$

where

$$\bar{\mathcal{D}} \equiv -(K^{-1} \Delta) \Phi_A^* \frac{\partial^l}{\partial \Phi_A^*} + (-)^{\epsilon_A} (D^{-1} \Delta)^{AB} \left(\frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B} \frac{\partial^r}{\partial \Phi^A} + \frac{1}{2} \frac{\partial^l \partial^r}{\partial \Phi^B \partial \Phi^A} \right). \quad (2.32)$$

§3. QM operator and anomalies

After these preparations, we may describe the main subject of the present paper. More results supporting the following arguments will be presented in later sections.

The QM operator is a composite operator

$$-\Lambda \frac{\partial}{\partial \Lambda} \bar{\Sigma}_\Lambda = \bar{\mathcal{D}} \bar{\Sigma}_\Lambda. \quad (3.1)$$

Thus, if the QM operator vanishes at some scale, it does so all down to $\Lambda = 0$: this is the manifestation of a symmetry. For an anomalous theory, however, it does not vanish and its asymptotic form in the limits of $\Lambda \rightarrow \infty$ after taking $\Lambda_0 \rightarrow \infty$ is an anomaly multiplied by a ghost, which will be denoted as $\mathcal{A}[\phi]$

$$\lim_{\Lambda \rightarrow \infty} \lim_{\Lambda_0 \rightarrow \infty} \bar{\Sigma}_\Lambda = \mathcal{A}[\phi]. \quad (3.2)$$

Note that $\mathcal{A}[\phi]$ is written in terms of the bare field ϕ . We will come back to Eq. (3.2) in a concrete example in the next subsection.

Under the quantum BRST transformation, the QM operator vanishes at any scale

$$\delta_Q \bar{\Sigma}_\Lambda = 0. \quad (3.3)$$

The QM operator is a cohomologically closed operator. This is an algebraic relation that holds even if the QM operator does not vanish.

By Eq. (2.25), we defined the 1PI counterpart for the QM operator, which has the expression as in Eq. (2.26). In the limit of $\Lambda \rightarrow 0$, we find Eq. (2.27). Therefore, for an anomalous theory, we would find

$$\bar{\Sigma}_B^{1PI} \equiv \lim_{\Lambda \rightarrow 0} \bar{\Sigma}_{B,\Lambda}^{1PI} = \frac{\partial^r \bar{\Gamma}_B}{\partial \varphi^A} \frac{\partial^l \bar{\Gamma}_B}{\partial \phi_A^*} = \mathcal{A}[\varphi]. \quad (3.4)$$

Here, on the r.h.s. of (3.4), there appears the same functional \mathcal{A} as Eq. (3.2), but written in terms of the classical field φ . In this limit, the cohomological condition on $\mathcal{A}[\varphi]$ is the Wess-Zumino condition,

$$(\mathcal{A}[\varphi], \bar{I}_B)_{\varphi, \phi^*} = 0 . \quad (3.5)$$

Let us further study the QM operator for finite Λ . We will give an argument for having the same functional in Eqs. (3.2) and (3.4).

As explained in Appendix D of Ref. 5), there holds the relation between $\bar{\Sigma}_\Lambda$ and $\bar{\Sigma}_{B,\Lambda}^{*,*}$)

$$\bar{\Sigma}_\Lambda[\Phi, \Phi^*] = \bar{\Sigma}_{B,\Lambda}^{1PI}[\varphi_\Lambda, \phi^*] , \quad (3.6)$$

$$\varphi_\Lambda^A = \frac{K_0}{K} \Phi^A + (K_0 - K)(D^{-1})^{AB} \frac{\partial^l \bar{S}_\Lambda}{\partial \Phi^B} . \quad (3.7)$$

In the presence of an anomaly, we may regard the QM operator as the composite operator, which becomes the anomaly multiplied by the ghost in both the UV and IR limits. Equation (3.6) tells us that the operator is a functional of φ_Λ and ϕ^* , where φ_Λ is a composite operator by itself. Therefore, we may write the QM operator as

$$\bar{\Sigma}_\Lambda = \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] . \quad (3.8)$$

The scale dependence of the operator originates from φ_Λ , as well as the scale dependence of coefficients. The latter scale dependence is expressed by the last Λ on the r.h.s. of Eq. (3.8).

We consider the flow equation for $\bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda]$,

$$-\Lambda \frac{\partial}{\partial \Lambda} \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] = \mathcal{D} \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] , \quad (3.9)$$

where

$$\mathcal{D} \equiv (D^{-1} \Delta)^{AB} \left(\frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B} \frac{\partial^l}{\partial \Phi^A} + \frac{1}{2} \frac{\partial^l}{\partial \Phi^B} \frac{\partial^l}{\partial \Phi^A} \right) . \quad (3.10)$$

The operator (3.10) is different from the one in Eq. (2.32) in two points: 1) in Eq. (3.9), we do not include the trivial scale change of the antifield given in Eq. (2.14); 2) the right derivative w.r.t. Φ^A in (2.32) is rewritten into the left derivative in (3.10).

Note that φ_Λ is a functional of Φ and ϕ^* via Eq. (3.7) and a composite operator by itself that follows the same flow equation (3.9) as $\bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda]$. Using this fact, we may separate the scale dependence of $\bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda]$ into two parts from φ_Λ and coefficients, respectively. The latter follows the equation

$$\left(-\Lambda \frac{\partial}{\partial \Lambda} \right)' \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] = \mathcal{D}' \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] , \quad (3.11)$$

where

$$\mathcal{D}' \equiv \frac{1}{2} (-)^{\epsilon_A + \epsilon_B(\epsilon_A + \epsilon_C)} (D^{-1} \Delta)^{AB} \left(\frac{\partial^l \varphi_\Lambda^C}{\partial \Phi^A} \frac{\partial^l \varphi_\Lambda^D}{\partial \Phi^B} \right) \frac{\partial^l}{\partial \varphi_\Lambda^D} \frac{\partial^l}{\partial \varphi_\Lambda^C} . \quad (3.12)$$

The prime on the derivative on the l.h.s. of Eq. (3.11) implies that it acts on the explicit scale dependence through coefficients.

In deriving Eq. (3.11), we used only the relation $\bar{\mathcal{O}}[\Phi, \Phi^*] = \bar{\mathcal{O}}^{1PI}[\varphi_\Lambda, \phi^*]$ for a composite operator and its 1PI counterpart. Therefore, it is valid for any 1PI composite operator.

Now, we make the loop expansion of $\bar{\mathcal{A}}$. Since there is no tree-level contribution, we find

$$\left(-\Lambda \frac{\partial}{\partial \Lambda} \right)' \bar{\mathcal{A}}^{(1)}[\varphi_\Lambda, \phi^*; \Lambda] = 0 \quad (3.13)$$

*) Equations (3.6) and (3.7) are different from Eqs. (D.16) and (D.19) of Ref. 5) in two points: 1) here, the UV cutoff Λ_0 is finite; 2) we rescaled the antifields according to Eq. (2.14) to respect the canonical relation. Other than these two minor differences, the relations are the same.

for the one-loop contribution. In other words, at the one-loop level, the scale dependence originates solely from φ_Λ .

Let us assume for the moment that the one-loop calculation is exact. In the limit of $\Lambda \rightarrow 0$, the operator becomes $\bar{\mathcal{A}}[\varphi, \phi^*]$ that must be cohomologically equivalent to the well-known form of the anomaly denoted as $\mathcal{A}[\varphi]$ earlier in Eq. (3.4). At this point, we realize that there must be a composite operator that tends to $\mathcal{A}[\varphi]$ in $\Lambda \rightarrow 0$. It is the operator $\mathcal{A}[\varphi_\Lambda]$ that is the same functional as Eq. (3.4) with all the fields replaced by φ_Λ .

We may summarize our discussion that all the known facts are consistent with the following expression for the QM operator,

$$\bar{\Sigma}_\Lambda[\Phi, \Phi^*] = \mathcal{A}[\varphi_\Lambda] . \quad (3.14)$$

Via the composite operator $\mathcal{A}[\varphi_\Lambda]$, the two limits (3.2) and (3.4) are related.

3.1. Anomaly and QM operator: $U(1)_V \times U(1)_A$ gauge theory

Here, we explain how the QM operator is related to an anomaly by taking the $U(1)_V \times U(1)_A$ gauge theory as an example.

Let us state a few facts that will be found useful later.

In general, the QM operator $\bar{\Sigma}_\Lambda[\Phi, \Phi^*]$ is related to the Ward-Takahashi operator $\Sigma_\Lambda[\Phi]$ as

$$\Sigma_\Lambda[\Phi] = \bar{\Sigma}_\Lambda[\Phi, \Phi^*]|_{\Phi^*=0}.$$

For QED, antifields appear in the Wilson action in a simple manner.^{10),11)} As a result, the QM operator may be obtained via shifting the fields in the WT operator:

$$\bar{\Sigma}_\Lambda = \Sigma_\Lambda[A_\mu^{sh}, c, \psi^{sh}, \bar{\psi}^{sh}], \quad (3.15)$$

as explained in Refs. 5) and 12). Here the superscript “*sh*” implies that they are shifted by terms with antifields. In other words, the antifield dependence of the QM operator appears only in these shifts. The shifted variables are^{*)}

$$\begin{aligned} A_\mu^{sh}(k) &\equiv A_\mu(k) - \frac{1 - K(k)}{k^2} k_\mu \bar{c}^*(k) , \\ \psi^{sh}(p) &\equiv \psi(p) + \frac{1 - K(p)}{\not{p} + im} e \int_k c(k) \psi^*(p - k) , \\ \bar{\psi}^{sh}(-p) &\equiv \bar{\psi}(-p) + e \int_k \bar{\psi}^*(-p - k) c(k) \frac{1 - K(p)}{\not{p} + im} . \end{aligned} \quad (3.16)$$

For later discussion, the form of the shifted gauge field will be of particular importance: the second term on the r.h.s. is proportional to the momentum k_μ . The origin of this term is the BRST transformation and $\phi^* \cdot \delta\phi$ term in the extended action.

Now, let us consider a $U(1)_V \times U(1)_A$ gauge theory with massless fermion with couplings

$$\int d^4x \bar{\psi} (e_V \not{A} + e_A \gamma_5 \not{B}) \psi.$$

For two gauge symmetries, we have WT operators, Σ_Λ^V and Σ_Λ^A . In Ref. 5), their asymptotic behaviours in $\Lambda \rightarrow \infty$ were studied. If we keep the vector gauge symmetry intact, $\Sigma_\Lambda^V = 0$, we find that the WT operator for the axial symmetry behaves as

$$\begin{aligned} \Sigma_\Lambda^A &\rightarrow -\frac{e_A e_V^2}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu A_\nu(k) q_\rho A_\sigma(q) \\ &\quad - \frac{e_A^3}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu B_\nu(k) q_\rho B_\sigma(q) \end{aligned} \quad (3.17)$$

^{*)} The expressions of the shifted variables are given in the limit of $\Lambda_0 \rightarrow \infty$. In the rest of this section, we assume that this limit has been taken.

in the limit of $\Lambda \rightarrow \infty$. Here c_A is the ghost field associated with the axial gauge symmetry.

Now, we consider the QM operator, $\bar{\Sigma}_A^A$. If one recalls the reason why we find the shift in the gauge field as Eq. (3-16) for QED, we understand that the same reason applies here for both gauge fields, and $\bar{\Sigma}_A^A$ is written in terms of shifted gauge fields,

$$\begin{aligned} A_\mu^{sh}(k) &\equiv A_\mu(k) - \frac{1 - K(k)}{k^2} k_\mu \bar{c}_V^*(k) , \\ B_\mu^{sh}(k) &\equiv B_\mu(k) - \frac{1 - K(k)}{k^2} k_\mu \bar{c}_A^*(k) . \end{aligned}$$

The shift parts, however, vanish in $\Lambda \rightarrow \infty$. We conclude that the QM operator has the same asymptotic form (3-17) as the WT operator.

§4. Anomaly via ERG calculation

Using the ERG approach, we explicitly calculate the anomaly contribution to the WT operator for an abelian gauge symmetry. We will understand where to find anomalous contributions. The calculations to determine counter terms will be omitted. It is also possible to extend the following calculations to non-abelian gauge symmetries.¹³⁾

First let us sketch our calculation. The WT operator Σ_A takes the following form

$$\Sigma_A = \bar{\Sigma}_A|_{\Phi^*=0} = \frac{\partial^r S_A}{\partial \Phi^A} \delta \Phi^A + \frac{\partial^l}{\partial \Phi^A} \delta \Phi^A \quad (4.1)$$

where $\delta \Phi^A \equiv (\Phi^A, \bar{S}_A)|_{\Phi^*=0}$. We will find the second term in (4.1) contains the fermion loop. After writing the one-loop contributions, we take $\Lambda_0 \rightarrow \infty$ and then $\Lambda \rightarrow \infty$. This procedure produces an anomaly times appropriate ghost.

To evaluate the WT operator, we need to know how the BRST transformation changes under the scale change. For a particular class of BRST transformation

$$\delta \phi^A = K_0 \left(\mathcal{R}_B^{(1)A}(\Lambda_0) \phi^B + \frac{1}{2} \mathcal{R}_{BC}^{(2)A}(\Lambda_0) \phi^B \phi^C \right)$$

we have

$$\delta \Phi^A = K \left(\mathcal{R}_B^{(1)A}(\Lambda_0) [\Phi^B]_\Lambda + \frac{1}{2} \mathcal{R}_{BC}^{(2)A}(\Lambda_0) [\Phi^B \Phi^C]_\Lambda \right) \quad (4.2)$$

for a lower scale Λ .¹⁴⁾ The $[\Phi^B]_\Lambda$ and $[\Phi^B \Phi^C]_\Lambda$ are the composite operators at the scale Λ :

$$[\Phi^A]_\Lambda \equiv \Phi^A + (K_0 - K)(D^{-1})^{AB} \frac{\partial^l S_{I,\Lambda}}{\partial \Phi^B} , \quad (4.3)$$

$$\begin{aligned} [\Phi^A \Phi^B]_\Lambda &\equiv [\Phi^A]_\Lambda [\Phi^B]_\Lambda \\ &+ (K_0 - K)(D^{-1})^{AC} (K_0 - K)(D^{-1})^{BD} \frac{\partial^l \partial^l S_I}{\partial \Phi^C \partial \Phi^D} . \end{aligned} \quad (4.4)$$

Let us take again the $U(1)_V \times U(1)_A$ gauge theory as our example. The interaction part of the classical action is

$$\mathcal{S}_I[\phi] = \int_{p, k} \bar{\psi}(-p-k) \left\{ e_V \not{A}(k) + e_A \gamma_5 \not{B}(k) \right\} \psi(p) , \quad (4.5)$$

and the classical BRST transformations of fermions and gauge fields are

$$\begin{aligned} \delta_V \psi(p) &= -e_V \int_k \psi(p-k) c_V(k), \quad \delta_V \bar{\psi}(-p) = e_V \int_k \bar{\psi}(-p-k) c_V(k), \\ \delta_V A_\mu(p) &= -p_\mu c_V(p), \quad \delta_V B_\mu(p) = 0, \end{aligned} \quad (4.6)$$

for the vector gauge symmetry and

$$\begin{aligned}\delta_A \psi(p) &= e_A \int_k \gamma_5 \psi(p-k) c_V(k), \quad \delta_A \bar{\psi}(-p) = e_A \int_k \bar{\psi}(-p-k) c_A(k) \gamma_5, \\ \delta_A A_\mu(p) &= 0, \quad \delta_A B_\mu(p) = -p_\mu c_A(p),\end{aligned}\tag{4.7}$$

for the axial gauge symmetry.

Since transformations in Eqs. (4.6) and (4.7) are bilinear in fields, we will have composite operators of the type (4.4). However, the symmetries are abelian; ghosts do not interact with other fields. Therefore, field transformations will be written with the fermion composite operators. Let us take the first transformation of (4.6) for example. According to Eq. (4.2), the transformation at the scale Λ is

$$\delta_V \Psi(p) = K \mathcal{R}^{(2)}(\Lambda_0) \int_k [\Psi]_\Lambda(p-k) C_V(k),\tag{4.8}$$

where

$$[\Psi]_\Lambda(p) = \Psi(p) + \frac{K_0 - K}{\not{p}} \frac{\partial^l S_{I,\Lambda}}{\partial \bar{\Psi}(-p)}.\tag{4.9}$$

To write down the interaction action $S_{I,\Lambda}$, we use interactions in S_{I,Λ_0} and integrate over fields with momenta between Λ_0 and Λ . The UV action S_{Λ_0} contains counter terms that also affect the coefficient $\mathcal{R}^{(2)}(\Lambda_0)$ in (4.8) as well. Here, as the lowest order calculations, we set $\Lambda_0 = \infty$ and ignore contributions from counter terms; we assume the classical value $-e_V$ for $\mathcal{R}^{(2)}(\Lambda_0)$.*)

Let us write down the interaction action at the second order in couplings that are relevant for calculating the anomaly,

$$\begin{aligned}S_{I,\Lambda}[\Phi] &= \\ &\frac{e_V^2}{2} \int_{l,k,q} \bar{\Psi}(-l-k) \left[A(k) \frac{(1-K)(l)}{\not{l}} A(q) + A(q) \frac{(1-K)(l+k-q)}{\not{l} + \not{k} - \not{q}} A(k) \right] \Psi(l-q) \\ &+ \frac{e_A^2}{2} \int_{l,k,q} \bar{\Psi}(-l-k) \left[B(k) \frac{(1-K)(l)}{\not{l}} B(q) + B(q) \frac{(1-K)(l+k-q)}{\not{l} + \not{k} - \not{q}} B(k) \right] \Psi(l-q) \\ &+ \frac{e_V e_A}{2} \int_{l,k,q} \bar{\Psi}(-l-k) \left[\gamma_5 B(k) \frac{(1-K)(l)}{\not{l}} A(q) + A(q) \frac{(1-K)(l+q-k)}{\not{l} + \not{k} - \not{q}} \gamma_5 B(k) \right. \\ &\quad \left. + A(k) \frac{(1-K)(l)}{\not{l}} \gamma_5 B(q) + \gamma_5 B(q) \frac{(1-K)(l+q-k)}{\not{l} + \not{k} - \not{q}} A(k) \right] \Psi(l-q).\end{aligned}\tag{4.10}$$

The second term of Eq. (4.1) with Eqs. (4.9) and (4.10) produces the fermion one-loop contribution to the WT operator, to be denoted $\Sigma_A^{(1)}$ in the following.

$$\Sigma_A^{V(1)} = e_V \int_{p,k} \text{tr} \left[\frac{\partial^l \partial^r S_{I,\Lambda}}{\partial \bar{\psi}(-p+k) \partial \psi(p)} U_V(-p, p-k) \right] c_V(k),\tag{4.11}$$

$$\Sigma_A^{A(1)} = -e_A \int_{p,k} \text{tr} \left[\frac{\partial^l \partial^r S_{I,\Lambda}}{\partial \bar{\psi}(-p+k) \partial \psi(p)} U_A(-p, p-k) \right] c_A(k),\tag{4.12}$$

where

$$\begin{aligned}U_V(-p, p-k) &\equiv \frac{K(p)(1-K)(p-k)}{\not{p} - \not{k}} - \frac{K(p-k)(1-K)(p)}{\not{p}}, \\ U_A(-p, p-k) &\equiv -U_V(-p, p-k) \gamma_5.\end{aligned}$$

Rewriting (4.12), we find

$$\begin{aligned}\Sigma_A^{A(1)} &= -e_A e_V^2 \int_{q,k} c_A(-q-k) \epsilon_{\mu\nu\rho\sigma} k_\mu A_\nu(k) A_\sigma(q) I_\rho(q, k) \\ &\quad - e_A^3 \int_{q,k} c_A(-q-k) \epsilon_{\mu\nu\rho\sigma} k_\mu B_\nu(k) B_\sigma(q) I_\rho(q, k),\end{aligned}\tag{4.13}$$

*) The calculation⁵⁾ quoted in the previous subsection is based on the asymptotic UV behaviours of the action. Therefore, contributions of counter terms are properly taken care of.

where $I_\rho(q, k)$ stands for the integral over $P = p + q$

$$I_\rho(q, k) = \int_P P_\rho K(P - q) \frac{(1 - K(P))(1 - K(P + k))}{P^2(P + k)^2}, \quad (4.14)$$

which can be evaluated for $\Lambda \gg q, k$. Expanding in the external momenta, we find in the cutoff-removed limit $\Lambda \rightarrow \infty$

$$I_\rho(q, k) \rightarrow q_\rho \int_P (-2) P^2 \frac{dK}{dP^2} \frac{(1 - K(P))^2}{P^4} = \frac{q_\rho}{24\pi^2}. \quad (4.15)$$

Here, use has been made of the integration formula over $\bar{q} = q/\Lambda$

$$\int_{\bar{q}} \frac{\Delta(\bar{q}^2)(1 - \kappa(\bar{q}^2))^n}{\bar{q}^4} = \frac{2}{(4\pi)^2} \frac{1}{n+1}, \quad (4.16)$$

which can be proved easily. $\Delta(\bar{q}^2)$ is defined in (2.29). Finally, we obtain

$$\begin{aligned} \Sigma_\infty^{A(1)} &= -\frac{e_A e_V^2}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu A_\nu(k) q_\rho A_\sigma(q) \\ &\quad - \frac{e_A^3}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu B_\nu(k) q_\rho B_\sigma(q) \\ &= \frac{e^2}{48\pi^2} \int_x c_A(x) \epsilon_{\mu\nu\rho\sigma} (F_{\mu\nu}^V(x) F_{\rho\sigma}^V(x) + F_{\mu\nu}^A(x) F_{\rho\sigma}^A(x)). \end{aligned} \quad (4.17)$$

Similarly, we find

$$\Sigma_\infty^{V(1)} = -2 \times \frac{e_A e_V^2}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_V(-q-k) k_\mu B_\nu(k) q_\rho A_\sigma(q). \quad (4.18)$$

We may add the following counter term to the Wilson action, $S_{I,\Lambda} \rightarrow S_{I,\Lambda} + S_c$:

$$S_c = a \epsilon_{\mu\nu\rho\sigma} \int_{q,k} B_\mu(k) A_\nu(-k-q) q_\rho A_\sigma(q), \quad (4.19)$$

where a is a constant to be determined below. The BRST transformations of the counter term are given as

$$\begin{aligned} \delta_V S_c &= -a \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_V(-q-k) k_\mu B_\nu(k) q_\rho A_\sigma(q), \\ \delta_A S_c &= a \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu A_\nu(k) q_\rho A_\sigma(q). \end{aligned} \quad (4.20)$$

Therefore, inclusion of the counter term $S_{I,\Lambda} \rightarrow S_{I,\Lambda} + S_c$ gives new contributions in $\Sigma_\infty^{A(1)}$ and $\Sigma_\infty^{V(1)}$ proportional to a :

$$\begin{aligned} \Sigma_\infty^{A(1)} &\rightarrow \Sigma_\infty^{A(1)} = \left(a - \frac{e_A e_V^2}{12\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu A_\nu(k) q_\rho A_\sigma(q) \\ &\quad - \frac{e_A^3}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu B_\nu(k) q_\rho B_\sigma(q) \\ \Sigma_\infty^{V(1)} &\rightarrow \Sigma_\infty^{V(1)} = -\left(a + \frac{e_A e_V^2}{6\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_V(-q-k) k_\mu B_\nu(k) q_\rho A_\sigma(q) \end{aligned} \quad (4.21)$$

We choose the parameter a as

$$a = -\frac{e_A e_V^2}{6\pi^2}, \quad (4.22)$$

so that the vector gauge symmetry is preserved. The anomaly for the axial gauge symmetry is changed to

$$\begin{aligned} \Sigma_\infty^{A(1)} = & -\frac{e_A e_V^2}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu A_\nu(k) q_\rho A_\sigma(q) \\ & -\frac{e_A^3}{12\pi^2} \epsilon_{\mu\nu\rho\sigma} \int_{q,k} c_A(-q-k) k_\mu B_\nu(k) q_\rho B_\sigma(q) . \end{aligned} \quad (4.23)$$

The result coincides with (3.17).

§5. Summary and discussion

We have argued how the QM operator may be regarded as the anomaly composite operator. The operator simplifies in both ends $\Lambda \rightarrow 0$ and ∞ : in an intermediate scale, a composite operator would consist of various operators if written in terms of $\bar{\Phi}$. We have further argued that it can be written as $\mathcal{A}[\varphi_\Lambda]$ for any scale if the one-loop calculation is exact. In the next subsection, we showed the validity of Eq. (3.2) for an abelian theory. We also presented a one-loop calculation of anomaly in §4.

The QM operator satisfies the algebraic condition $\delta_Q \bar{\Sigma}_\Lambda = 0$. Obviously, this tells us that, at any scale of the cutoff, an anomaly is a closed form that provides us a nontrivial element of the BRST cohomology. Since we have the relation (3.6), $\bar{\Sigma}_\Lambda^{1PI}$ also satisfies the same condition. However, writing it in terms of the effective average action would not give an illuminating condition. Observe that with a finite cutoff Λ , $\bar{\Sigma}_\Lambda^{1PI}$ itself is not particularly simple. Only in the limit of $\Lambda \rightarrow 0$, we can show

$$\left(\frac{\partial^r \bar{\Gamma}_B}{\partial \varphi^A} \frac{\partial^l \bar{\Gamma}_B}{\partial \phi_A^*}, \bar{\Gamma}_B \right)_{\varphi, \phi^*} = e^{-\bar{W}_B} \int \mathcal{D}\phi \left(\delta_Q \bar{\Sigma}_B \right) e^{\bar{S}_B + K_0^{-1} J \cdot \phi} \quad (5.1)$$

after a straightforward but lengthy calculation explained in the Appendix. The Wess-Zumino condition on the l.h.s. is related to the condition on $\bar{\Sigma}_B$.

In earlier works,^{(15)–(17)} anomalies have been calculated in ERG approaches. Although the formulations are different from ours, it was pointed out that anomalies appear in asymptotic behaviours of operators related to the WT and QM operators in our terminology. The authors of Ref. 16) studied non-abelian anomaly including the evaluation of necessary counter terms. The advantage of our formulation is the algebraic structure of the antifield formalism. That made our discussion more transparent.

In the context of the renormalization group, several proofs^{(18), (19)} were given for the non-renormalization theorem. Addressing the theorem in the present framework of ERG is an important and interesting question. We leave it for future work.

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Appendix A

—— Proof of Eq. (5.1) ——

We will show the following relation for finite cutoffs Λ and Λ_0 :

$$\left(\mathcal{A}_{B,\Lambda}, \bar{\Gamma}_{B,\Lambda} \right)_{\varphi_\Lambda, \phi^*} = e^{-\bar{W}_{B,\Lambda}} \int \mathcal{D}\phi \left(\delta'_Q \bar{\Sigma}_{B,\Lambda} \right) e^{\bar{S}_{B,\Lambda} + K_0^{-1} J \cdot \phi} . \quad (A.1)$$

Let us explain the notations. $\mathcal{A}_{B,\Lambda}$ stands for the quantity

$$\mathcal{A}_{B,\Lambda} \equiv \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_A^A} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} = e^{-\bar{W}_{B,\Lambda}} \int \mathcal{D}\phi \bar{\Sigma}_{B,\Lambda} e^{\bar{S}_{B,\Lambda} + K_0^{-1} J \cdot \phi} , \quad (A.2)$$

where the second equality was shown in Ref. 5). δ'_Q and $\bar{\Sigma}_{B,\Lambda}$ are the BRST transformation and QM operator defined with the action $\bar{S}_{B,\Lambda}$ respectively:

$$\delta'_Q X \equiv (X, \bar{S}_{B,\Lambda}) + \Delta X, \quad (\text{A}\cdot 3)$$

$$\bar{\Sigma}_{B,\Lambda} \equiv \frac{1}{2}(\bar{S}_{B,\Lambda}, \bar{S}_{B,\Lambda}) + \Delta \bar{S}_{B,\Lambda}. \quad (\text{A}\cdot 4)$$

It is clear from Eq. (2·20) that the difference between \bar{S}_B and $\bar{S}_{B,\Lambda}$ vanishes in $\Lambda \rightarrow 0$. Therefore, in this limit,

$$\delta'_Q \rightarrow \delta_Q, \quad \bar{\Sigma}_{B,\Lambda} \rightarrow \bar{\Sigma}_B.$$

By definitions given in Eqs. (2·18) and (2·22), we also have

$$\bar{W}_{B,\Lambda} \rightarrow \bar{W}_B, \quad \bar{\Gamma}_{B,\Lambda} \rightarrow \bar{\Gamma}_B \quad (\text{A}\cdot 5)$$

in the same limit. Sending $\Lambda \rightarrow 0$ in Eq. (A·1), we find Eq. (5·1).

Now, we give a proof of (A·1). The bracket on the l.h.s. is defined with respect to φ_Λ and ϕ^* ,

$$(\mathcal{A}_{B,\Lambda}, \bar{\Gamma}_{B,\Lambda})_{\varphi_\Lambda, \phi^*} = \left. \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial \varphi_\Lambda^B} \right|_{\phi^*} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_B^*} - \left. \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial \phi_B^*} \right|_{\varphi_\Lambda} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B}. \quad (\text{A}\cdot 6)$$

Using the last expression of Eq. (A·2), we may regard $\mathcal{A}_{B,\Lambda}$ as a functional of J and ϕ^* , $\mathcal{A}_{B,\Lambda} = \mathcal{A}_{B,\Lambda}(J, \phi^*)$. Since the source J^A is a functional of φ_Λ and ϕ^* via the relation

$$J^A = -K_0 \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^A}, \quad (\text{A}\cdot 7)$$

$\mathcal{A}_{B,\Lambda}$ depends on φ_Λ and ϕ^* as

$$\mathcal{A}_{B,\Lambda} = \mathcal{A}_{B,\Lambda}(J(\varphi_\Lambda, \phi^*), \phi^*). \quad (\text{A}\cdot 8)$$

Therefore, we find

$$\begin{aligned} (\mathcal{A}_{B,\Lambda}, \bar{\Gamma}_{B,\Lambda})_{\varphi_\Lambda, \phi^*} &= \left(\mathcal{A}_{B,\Lambda}(J(\varphi_\Lambda, \phi^*), \phi^*), \bar{\Gamma}_{B,\Lambda} \right)_{\varphi_\Lambda, \phi^*} \\ &= \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial J^A} (J^A, \bar{\Gamma}_{B,\Lambda})_{\varphi_\Lambda, \phi^*} - \left. \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial \phi_B^*} \right|_J \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B} \\ &= - \left. \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial \phi_B^*} \right|_J \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B}, \end{aligned} \quad (\text{A}\cdot 9)$$

since the first term of the second line vanishes

$$\frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial J^A} (J^A, \bar{\Gamma}_{B,\Lambda})_{\varphi_\Lambda, \phi^*} = K_0^2 (-)^{\epsilon_A \epsilon_B} \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial J^A} \frac{\partial^r \mathcal{A}_{B,\Lambda}}{\partial J^B} (\bar{\Gamma}_{B,\Lambda}^{(2)})_{AB} = 0, \quad (\text{A}\cdot 10)$$

where

$$(\bar{\Gamma}_{B,\Lambda}^{(2)})_{AB} \equiv \frac{\partial^l}{\partial \varphi_\Lambda^A} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B}.$$

Equation (A·10) is easily understood once we notice the following symmetric properties,

$$\epsilon(\bar{\Gamma}_{AC}^{(2)}) = \epsilon_A + \epsilon_C, \quad \bar{\Gamma}_{CA}^{(2)} = (-)^{\epsilon_A + \epsilon_C + \epsilon_A \epsilon_C} \bar{\Gamma}_{AC}^{(2)}.$$

In calculating the ϕ^* -derivative of $\mathcal{A}_{B,\Lambda}$ on the r.h.s. of Eq. (A·9), we use the path integral expression in Eq. (A·2). The derivative acting on the factor $\exp(-\bar{W}_{B,\Lambda})$ produces the term

$$\begin{aligned} \frac{\partial^l \bar{W}_{B,\Lambda}}{\partial \phi_B^*} \mathcal{A}_{B,\Lambda} \cdot \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B} &= \mathcal{A}_{B,\Lambda} (-)^{\epsilon_A + 1} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B} \frac{\partial^l \bar{W}_{B,\Lambda}}{\partial \phi_B^*} \\ &= -\mathcal{A}_{B,\Lambda} \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^B} \frac{\partial^l \bar{W}_{B,\Lambda}}{\partial \phi_B^*} = -\mathcal{A}_{B,\Lambda}^2 = 0. \end{aligned} \quad (\text{A}\cdot 11)$$

Here, use was made of the relation in the second line,

$$\frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} = \frac{\partial^l \bar{W}_{B,\Lambda}}{\partial \phi_A^*} + \frac{\partial^l \bar{J}_C}{\partial \phi_A^*} \left(\frac{\partial^l \bar{W}_{B,\Lambda}}{\partial J^C} - K_0^{-1} \varphi_A^C \right) = \frac{\partial^l \bar{W}_{B,\Lambda}}{\partial \phi_A^*}.$$

Thus, we may only consider the ϕ^* -derivative of the expression under the path integral in Eq. (A·2).

$$\begin{aligned} (\mathcal{A}_{B,\Lambda}, \bar{\Gamma}_{B,\Lambda})_{\varphi_A, \phi^*} &= -e^{-\bar{W}_{B,\Lambda}} \frac{\partial^r}{\partial \phi_B^*} \left(\int \mathcal{D}\phi \bar{\Sigma}_{B,\Lambda} e^{\bar{S}_{B,\Lambda} + K_0^{-1} J \cdot \phi} \right) \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_B^*} (-)^{\epsilon_B} \\ &= e^{-\bar{W}_{B,\Lambda}} \frac{\partial^r}{\partial \phi_B^*} \left(\int \mathcal{D}\phi \bar{\Sigma}_{B,\Lambda} e^{\bar{S}_{B,\Lambda}} \frac{\partial^l}{\partial \phi_B} e^{K_0^{-1} J \cdot \phi} \right) \\ &= -(-)^{\epsilon_B} e^{-\bar{W}_{B,\Lambda}} \int \mathcal{D}\phi \left[\frac{\partial^r}{\partial \phi_B^*} \frac{\partial^l}{\partial \phi_B} \left(\bar{\Sigma}_{B,\Lambda} e^{\bar{S}_{B,\Lambda}} \right) \right] e^{K_0^{-1} J \cdot \phi}. \end{aligned} \quad (\text{A}\cdot 12)$$

In the second line of (A·12), we rewrote the factor $\frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_B^*}$ as the source J^B , then the field derivative under the path integral, and finally performed the partial integration. The quantity on the third line of (A·12) may be rewritten as

$$-(-)^{\epsilon_B} \left[\frac{\partial^r}{\partial \phi_B^*} \frac{\partial^l}{\partial \phi_B} \left(\bar{\Sigma}_{B,\Lambda} e^{\bar{S}_{B,\Lambda}} \right) \right] = e^{\bar{S}_{B,\Lambda}} (\delta_Q \bar{\Sigma}_{B,\Lambda} + \bar{\Sigma}_{B,\Lambda}^2). \quad (\text{A}\cdot 13)$$

Here, the second term on the r.h.s. vanishes since $\bar{\Sigma}_{B,\Lambda}$ is Grassmann odd. Substituting Eq. (A·13) to the last expression of Eq. (A·12), we reach the announced result (A·1).

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